SUMMARY
For various reasons, data are often transformed for analysis. It is often found that heritabilities and correlations are not greatly affected by transformation, though there are exceptions. When responses in different traits are to be compared, differences in scales should be accounted for. It is shown here that when responses are measured in standard deviation units there will usually be little effect of transformation, whereas if responses are measured in percentages of the mean the effect of transformation can be very large.

INTRODUCTION
“The scales of the instruments which we employ in measuring our plants and animals are those which experience has shown to be convenient to us. We have no reason to suppose that they are specially appropriate to the representation of the characters of a living organism for the purposes of genetical analysis” (Mather and Jinks, 1982, p. 63). It is not unusual for traits to be transformed for better analysis, and ASREML (Gilmour et al., 2002) for example includes the possibility of transforming input variables for analysis. Transformations may be used to stabilise variances across environments, to normalise residuals in order to satisfy statistical assumptions, to make genetic effects behave in an additive manner, or for other reasons. It is well recognised that the change of scale involved in transformations has effects on the interpretation of results. When selection responses in different traits are compared, some form of standardisation is required. Two such approaches are to measure response as a fraction of the original mean, and to measure it in standard deviation units. The aim of this paper is to compare the effects of transformation on these approaches.

THEORY
Suppose we are interested in a trait $X$ and possible transformations (such as square root or logarithm). We can write the value of the trait as $X = \bar{X} + \delta X$ where $\bar{X}$ is the mean and $\delta X$ is the deviation from the mean. Then the mean of $\delta X$ is zero and $E(\delta X)^2 = \sigma^2_x$.

Now suppose we transform using the function $f(X)$, where $f(\bar{X}) = f(\bar{X} + \delta X)$. Though the purpose of transformation is to use its non-linear qualities, we can get a first approximation to its properties by a linear approximation, taking just the first terms of its Taylor series expansion. Then $f(\bar{X} + \delta X) \approx f(\bar{X}) + f'(\bar{X})\delta X + \ldots$ where $f'(\bar{X}) = \frac{df}{d\bar{X}}$ evaluated at $\bar{X}$. We are ignoring terms in $(\delta X)^2$ and higher order. Then we have, since $E(\delta X) = 0$, the mean of transformed
values is \( f(\bar{X}) \) approximately, so that deviations from the mean are approximately \( f'(\bar{X}) \delta X \) and thus the standard deviation is \( f'(\bar{X}) \sigma_X \).

Now suppose that there is a change in mean from \( \bar{X} \) to \( \bar{X} + \Delta \). The fractional change in the mean is then \( \Delta / \bar{X} \), while the change as measured in standard deviation units is \( \Delta / \sigma_X \).

We can approximate the change in the transformed value as follows. \( f(\bar{X} + \Delta) \approx f(\bar{X}) + f'(\bar{X}) \Delta \), so that the change in the transformed values averages approximately \( f'(\bar{X}) \Delta \). Expressed in standard deviation units this is \( f'(\bar{X}) \Delta / f'(\bar{X}) \sigma_X \) or \( \Delta / \sigma_X \), the same as on the original scale. The fractional change is \( f'(\bar{X}) \Delta / f(\bar{X}) \). We can write this, on multiplying numerator and denominator by \( \bar{X} \), as \( \frac{\bar{X}f'(\bar{X}) \Delta}{f(\bar{X})} \) so that the ratio on the original scale, \( \Delta / \bar{X} \), is multiplied by \( \frac{\bar{X}f'(\bar{X})}{f(\bar{X})} \). This ratio will usually differ from unity.

Suppose we consider the square root transformation, \( f(X) = \sqrt{X} \). Then \( f'(X) = \frac{1}{2\sqrt{X}} \) and \( \frac{\bar{X}f'(\bar{X})}{f(\bar{X})} = \frac{\bar{X}/2\sqrt{\bar{X}}}{\sqrt{\bar{X}}} = \frac{1}{2} \). Thus we expect that the fractional or percentage change on the square root scale will be about half of that on the original scale.

Now consider the \( \log_{10} \) transformation. The logarithm to base 10 is 0.4343 times the natural logarithm, so that \( \log_{10}(X) = 0.4343 \ln(X) \). Thus what is true for the natural logarithm will also be true for the base 10 logarithm in the comparisons. Noting that when \( f(X) = \ln(X), f'(X) = 1/X \) we have \( \frac{\bar{X}f'(\bar{X})}{f(\bar{X})} = \frac{\bar{X}(1/\bar{X})}{\ln(\bar{X})} = 1/\ln(\bar{X}) \).

If we wish to be a little more precise, we can use the second degree terms in the Taylor series. This gives us the equation;

\[
f(X + \delta X) = f(X) + f'(X) \delta X + \left( \frac{1}{2} \right) f''(X) (\delta X)^2
\]

ignoring terms higher than the second degree. To this degree of accuracy, we have

\[
E(f(X) ) = f(X) + \left( \frac{1}{2} \right) f''(X) \sigma_X^2
\]
We then have as the deviation from the mean \( f'(\bar{X})\delta X + \left(\frac{1}{2}\right)f''(\bar{X})[(\delta X)^2 - \sigma_X^2] \) and we find our approximation to the variance by squaring this expression and taking expectations. This leads to

\[
\text{Var}[f(X)] \approx f'(\bar{X})^2 \sigma_X^2 + f'(\bar{X})f''(\bar{X})\mu_{3X} + \left(\frac{1}{4}\right)f''(\bar{X})^2[\mu_{4X} - \sigma_X^4]
\]

For a normal distribution of \( X \) the third moment is zero and the fourth moment is \( 3\sigma_X^4 \).

In this case the variance becomes \( f'(\bar{X})^2 \sigma_X^2 + \left(\frac{1}{2}\right)f''(\bar{X})^2 \sigma_X^4 \) approximately.

Now suppose there is a change \( \Delta \) in the mean of \( X \). The corresponding change in the mean of \( f(X) \) will be given by

\[
f'(\bar{X})\Delta + \left(\frac{1}{2}\right)f''(\bar{X})[\Delta^2 - \sigma_X^2]
\]

and may be greater than or less than the linear approximation.

Consider \( Y = \sqrt{X} \) and \( Z = \ln(X) \). Then replacing \( f \) by \( y \) or \( z \) as appropriate, we have

\[
y'(X) = \frac{1}{2}\sqrt{X}, \quad y''(X) = -\frac{1}{4}X^{-3/2} \\
z'(X) = \frac{1}{X}, \quad z''(X) = -\frac{1}{X^2}
\]

On writing \( C^2 = \sigma_X^2 / \bar{X}^2 \) we find that

\[
E(Y) = \sqrt{\bar{X}}[1 - C^2/8] \quad \text{and} \quad E(Z) = \ln(\bar{X}) - C^2/2
\]

This shows that it is the magnitude of \( C \) which determines how well the linear approximation works. For common values of \( C \) such as 0.2, this adjustment will be of very little importance. Thus although we can give more complex expressions using these second order terms, in most cases there is too little improvement to make this worthwhile.

**EXAMPLE**

As a check on the applicability of the above theory, a simulated sheep selection experiment was run. The trait selected had an initial mean of 10, a standard deviation of 1 and heritability 0.25. The line was selected for 20 years, and the pedigrees and trait values of the 3998 simulated sheep were output for analysis by ASREML. As well as on the original scale, the data were analysed using square root, \( \log_{10} \), square, and 100 times reciprocal transformations. From the ASREML output, the estimate of \( \mu \) was taken as the original mean, and the genetic gain (\( \Delta \)) as the mean EBV of the sheep in year 20. The genetic and phenotypic standard deviations and heritability were the usual ones.

**Table 1. Means, gains and ratios for different scales of measurement**

<table>
<thead>
<tr>
<th>Scale</th>
<th>( X )</th>
<th>( \sqrt{X} )</th>
<th>( \log_{10}X )</th>
<th>( X^2 )</th>
<th>( 100/X )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean ( (\mu) )</td>
<td>9.875</td>
<td>3.138</td>
<td>0.9922</td>
<td>98.49</td>
<td>10.24</td>
</tr>
<tr>
<td>Gain ( (\Delta) )</td>
<td>2.0829</td>
<td>0.3316</td>
<td>0.0934</td>
<td>40.88</td>
<td>-2.228</td>
</tr>
<tr>
<td>( \Delta/\mu )</td>
<td>0.2109</td>
<td>0.1057</td>
<td>0.0942</td>
<td>0.4151</td>
<td>-0.2234</td>
</tr>
<tr>
<td>Ratio</td>
<td>1</td>
<td>0.501</td>
<td>0.447</td>
<td>1.968</td>
<td>-1.059</td>
</tr>
<tr>
<td>Predicted</td>
<td>1</td>
<td>0.5</td>
<td>0.434</td>
<td>2</td>
<td>-1</td>
</tr>
</tbody>
</table>
Table 1 shows the estimates of means, Δ values and ratios, with the predicted values of the ratios using the true mean.

Table 2 shows the estimates of genetic and phenotypic standard deviations, heritabilities and ratios of gain to standard deviations.

**Table 2. Standard deviations, heritabilities and response ratios for different scales of measurement**

<table>
<thead>
<tr>
<th>Scale</th>
<th>X</th>
<th>√X</th>
<th>Log10X</th>
<th>X²</th>
<th>100/X</th>
</tr>
</thead>
<tbody>
<tr>
<td>σ_0</td>
<td>0.4642</td>
<td>0.0740</td>
<td>0.0207</td>
<td>9.3195</td>
<td>0.5091</td>
</tr>
<tr>
<td>σ_p</td>
<td>1.0111</td>
<td>0.1536</td>
<td>0.0407</td>
<td>22.3398</td>
<td>0.8924</td>
</tr>
<tr>
<td>h²</td>
<td>0.2108</td>
<td>0.2325</td>
<td>0.2595</td>
<td>0.1740</td>
<td>0.3254</td>
</tr>
<tr>
<td>SE(h²)</td>
<td>0.0337</td>
<td>0.0345</td>
<td>0.0356</td>
<td>0.0312</td>
<td>0.0373</td>
</tr>
<tr>
<td>Δ/σ_0</td>
<td>4.49</td>
<td>4.48</td>
<td>4.51</td>
<td>4.39</td>
<td>-4.49</td>
</tr>
<tr>
<td>Δ/σ_p</td>
<td>2.06</td>
<td>2.16</td>
<td>2.30</td>
<td>1.83</td>
<td>-2.56</td>
</tr>
</tbody>
</table>

It is obvious that the responses to selection expressed in standard deviation units are very little affected by the transformations, the similarity of the magnitudes of Δ/σ_0 being remarkable. There seems little doubt that the theory can be confidently applied in many cases in practice. There will be exceptions. A step function [=1 if X >11, =0 otherwise] was also used as a transformation. Since the step function has no derivative, the theory above does not apply. In this case ASREML analysis gave μ = 0.1411, Δ = 0.1966, σ_G = 0.0916, σ_p = 0.4369, with h² = 0.0440 ± 0.0192. In this case Δ/μ = 1.39, Δ/σ_G = 2.14 and Δ/σ_p = 0.45. These values are clearly different from those obtained with the other transformations.

It is obvious that the measurement of responses in (genetic or phenotypic) standard deviation units will very often be only minimally affected by the use of transformations, while standardisation in units of the original mean will produce very different results for different transformations. It needs to be remembered that some transformations will not behave so conveniently.

**ACKNOWLEDGMENTS**

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**REFERENCES**
